

# ON THE UNIVERSAL $\mathrm{CH}_0$ GROUP OF CUBIC THREEFOLDS IN POSITIVE CHARACTERISTIC

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**ABSTRACT.** We adapt for algebraically closed fields  $k$  of characteristic greater than 2 two results of Voisin, presented in [32], on the decomposition of the diagonal of a smooth cubic hypersurface  $X$  of dimension 3 over  $\mathbb{C}$ , namely: the equivalence between Chow-theoretic and cohomological decompositions of the diagonal of those hypersurfaces and the equivalence between the algebraicity (with  $\mathbb{Z}_2$ -coefficients) of the minimal class  $\theta^4/4!$  of the intermediate Jacobian  $J(X)$  of  $X$  and the cohomological (hence Chow-theoretic) decomposition of the diagonal of  $X$ . Using the second result, the Tate conjecture for divisors on surfaces defined over finite fields predicts, via a theorem of Schoen ([25]), that every smooth cubic hypersurface of dimension 3 over the algebraic closure of a finite field of characteristic  $> 2$  admits a Chow-theoretic decomposition of the diagonal.

## 1. INTRODUCTION

Let  $k$  be an algebraically closed field and  $X$  a smooth  $n$ -dimensional projective  $k$ -variety (irreducible).  $X$  is said to have universally trivial  $\mathrm{CH}_0$  group if for any field  $L$  containing  $k$ ,  $\mathrm{CH}_0(X_L) = \mathbb{Z}$ . As explained in [2],  $X$  has universally trivial  $\mathrm{CH}_0$  group if and only if  $\mathrm{CH}_0(X_{k(X)}) = \mathbb{Z}$ . We recall briefly the idea of the proof; it uses, for any extension  $k \subset L$ , the action, by the identity, of the correspondence  $\Delta_{X_L}$  on  $\mathrm{CH}_0(X_L)$  and the equivalence of the two previous properties with a so called Chow-theoretic decomposition of the diagonal. Passing to the limit in the diagonal morphism  $X \rightarrow X \times_k X$  over all open subsets  $V \subset X$  ( $V \rightarrow V \times_k V \hookrightarrow V \times_k X$ ) yields the diagonal point  $\delta_X \in X(k(X))$  (image of the generic point of  $X$  by the diagonal morphism). When  $\mathrm{CH}_0(X_{k(X)}) = \mathbb{Z}$ , the diagonal point is rationally equivalent over  $k(X)$  to a constant point  $x_{k(X)} = x \times_k k(X)$ , with  $x \in X(k)$ . Then, using the equalities  $\mathrm{CH}_0(X_{k(X)}) = \mathrm{CH}^n(X_{k(X)}) = \varinjlim_{U \subset X \text{ open}} \mathrm{CH}^n(U \times_k X)$  and the localization exact sequence, one gets an equality

$$(1) \quad \Delta_X = X \times_k x + Z \text{ in } \mathrm{CH}^n(X \times_k X)$$

where  $Z$  is supported on  $D \times_k X$  for some proper closed subset  $D$  of  $X$  i.e. a Chow theoretic decomposition of the diagonal. Now, letting  $L/k$  be an extension, a base change in (1) yields

$$\Delta_{X_L} = X_L \times_L x_L + Z_L \text{ in } \mathrm{CH}^n(X_L \times_L X_L)$$

so that letting both sides act on  $\mathrm{CH}_0(X_L)$  and using the fact that  $\Delta_{X_L}$  acts as the identity, one sees that  $\mathrm{CH}_0(X_L) = \mathbb{Z}$ . So, having a universally trivial  $\mathrm{CH}_0$  is equivalent to the existence of a Chow-theoretic decomposition of the diagonal. Projective spaces have universally trivial  $\mathrm{CH}_0$  so stably rational projective varieties also have universally trivial  $\mathrm{CH}_0$ . Voisin studied in the case  $k = \mathbb{C}$  the a priori weaker property of the existence of a cohomological decomposition of the diagonal

$$(2) \quad [\Delta_X] = [X \times_k x] + [Z] \text{ in } H_B^{2n}(X \times_k X, \mathbb{Z})$$

where  $Z \in \mathrm{CH}^n(X \times_k X)$  is supported on  $D \times_k X$  for some proper closed subset  $D$  of  $X$  and the cohomology is the Betti cohomology. In [32], she proved, in characteristic zero, that this weaker property is, in fact, equivalent to the existence of a Chow-theoretic decomposition of the diagonal for smooth cubic hypersurfaces of odd dimension ( $\geq 3$ ) or of dimension 4. She also worked out necessary and sufficient conditions for the existence of a cohomological (hence Chow-theoretic) decomposition of the diagonal for cubic threefolds, among other varieties. The second section of this paper is devoted to the proof of the equivalence between

cohomological and Chow-theoretic decomposition of the diagonal of a cubic threefold in positive characteristic, greater than 2:

**Theorem 1.1.** *Let  $k$  be an algebraically closed field of characteristic greater than 2 and  $X \subset \mathbb{P}_k^4$  be a smooth cubic hypersurface. Then  $X$  admits a Chow-theoretic decomposition of the diagonal (i.e. has universally trivial  $\mathrm{CH}_0$ ) if and only if it admits a cohomological decomposition of the diagonal with coefficients in  $\mathbb{Z}_2$ .*

**Remark 1.2.** (1) *In the Betti setting, having a cohomological decomposition of the diagonal with  $\mathbb{Z}$  coefficients is equivalent to having a cohomological decomposition of the diagonal with coefficients in  $\mathbb{Z}/2\mathbb{Z}$  since  $2\Delta_X$  has a Chow-theoretic decomposition (see Proposition 2.2). This is why in our case, only étale cohomology with  $\mathbb{Z}_2$  coefficients is used. In fact,  $\mathbb{Z}/2\mathbb{Z}$  would also do.*

(2) *Theorem 1.1 could have been stated also for cubic fourfolds and odd dimensional cubic of higher dimension since the proof given by Voisin adapts to the positive characteristic setting. But since in the case of cubic threefolds we can give a shorter proof and it is the only case where we have an interesting application in sight, we state the theorem only in this case.*

We also check that there is a criterion, similar to the one given in [32, Theorem 4.1] and [31, Theorem 4.9] in characteristic zero, for the existence of a cohomological decomposition of the diagonal of a cubic threefold.

**Theorem 1.3.** *Let  $X \subset \mathbb{P}_k^4$  be a smooth cubic hypersurface ( $k = \bar{k}$  and  $\mathrm{char}(k) > 2$ ). Then  $X$  admits a cohomological (hence Chow-theoretic) decomposition of the diagonal if and only if the principal polarization,  $\theta$ , of the associated Prym variety,  $J(X)$ , satisfies the following property: there is a cycle  $Z \in \mathrm{CH}_1(J(X)) \otimes \mathbb{Z}_2$  such that  $[Z] = \frac{[\theta]^4}{4!}$  in  $H^8(J(X), \mathbb{Z}_2)$ .*

Using Theorem 1.3 and a theorem of C. Schoen (see [6], [25]), we get the following consequence:

**Theorem 1.4.** *On an algebraic closure of a finite field of characteristic greater than 2, assuming the Tate conjecture for divisors on surfaces, every smooth cubic hypersurface of dimension 3 has universally trivial  $\mathrm{CH}_0$  group.*

In the Betti setting, a key feature in the proof of the criterion was the existence of a parametrization of the intermediate Jacobian of cubic threefolds with separably rationally connected general fiber, namely the condition:

(\*) there exist a smooth quasi-projective  $k$ -variety  $B$  and a correspondence  $Z \in \mathrm{CH}^2(B \times_k X)$  with  $Z_b \in \mathrm{CH}^2(X)$  trivial modulo algebraic equivalence for any  $b \in B(k)$  such that the induced Abel-Jacobi morphism  $\phi : B \rightarrow J(X)$  to the intermediate Jacobian  $J(X)$  of the cubic threefold  $X$  is dominant with  $\mathbb{P}^5$  as general fiber.

In Section 3, after a reminder on the definition of the intermediate Jacobian of a cubic threefold and Abel-Jacobi morphisms in the positive characteristic setting, we prove Theorem 1.3 and Theorem 1.4 under the assumption that (\*) is still true in our setting.

Over the complex numbers, such a parametrization was constructed by Iliev-Markushevich and Markushevich-Tikhomirov ([17] and [15], see also [9]) using the space of smooth normal elliptic quintics lying on the cubic hypersurface. Section 4 will be devoted to proving that we still have (\*) using the space of stable normal elliptic quintics.

For a variety  $X$ , the cohomology groups  $H^i(X, \mathbb{Z}_\ell)$  will be the étale cohomology groups and  $H_B^i(X, \mathbb{Z}_\ell)$ , if  $X$  is defined over a field  $K \hookrightarrow \mathbb{C}$  of characteristic 0, will be the Betti cohomology group of  $X_{\mathbb{C}}^{\mathrm{an}}$ .

Throughout this text,  $k$  will denote an algebraically closed field of characteristic  $> 2$ .

For a smooth projective  $k$ -variety  $X$ ,  $\langle \cdot, \cdot \rangle_X$  will denote the intersection pairing on  $H^*(X, \mathbb{Z}_\ell)$

induced by the cup-product and the trace map. We will use the following standard facts about cubic hypersurfaces in  $\mathbb{P}_k^4$ . Let  $X$  be a smooth cubic hypersurface in  $\mathbb{P}_k^4$ .

- (1) From the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_k^4}(-3) \rightarrow \mathcal{O}_{\mathbb{P}_k^4} \rightarrow \mathcal{O}_X \rightarrow 0$$

we have  $\omega_X \simeq \mathcal{O}_X(-2)$  and  $h^i(\mathcal{O}_X(k)) = 0$  for  $i \in \{1, 2\}$ ,  $k \in \mathbb{Z}$ .

- (2) The Fano variety of lines  $F(X) = \{[l] \in \mathrm{Gr}(2, 5), l \subset X\}$  of  $X$  is a smooth projective surface.

- (3) By Lefschetz hyperplane theorem, for  $\ell \neq p$  the  $\ell$ -adic cohomology of  $X$  is

$$H^0(X, \mathbb{Z}_\ell) = \mathbb{Z}_\ell \cdot [X], \quad H^2(X, \mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell \cdot [\mathcal{O}_X(1)], \quad H^4(X, \mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell \cdot [l], \quad H^6(X, \mathbb{Z}_\ell(3)) = \mathbb{Z}_\ell \cdot [x],$$

$$H^1(X, \mathbb{Z}_\ell) = 0 = H^5(X, \mathbb{Z}_\ell)$$

where  $[l]$  is the class of a line  $[l] \in F(X)$ .

- (4) Using, for example, a smooth proper lifting of  $X$  to characteristic zero ([18, Section 20]), we have  $H^3(X, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell^{10}$ . By Grothendieck-Lefschetz theorem,  $\mathrm{Pic}(X) \simeq \mathbb{Z} \cdot [\mathcal{O}_X(1)]$ .

## 2. CHOW-THEORETIC AND $\mathbb{Z}_2$ -COHOMOLOGICAL DECOMPOSITION OF THE DIAGONAL

In this section, for a  $k$ -variety  $Y$ ,  $B^i(Y)$  will designate the Chow group of codimension  $i$  cycle modulo algebraic equivalence. We prove in this section Theorem 1.1, adapting arguments of Voisin presented in [32] to the positive characteristic case. The key point is to prove that one can derive a decomposition of the diagonal modulo algebraic equivalence from a cohomological decomposition of the diagonal  $\Delta_X$  of a smooth cubic hypersurface  $X$ . Then we use the following proposition to obtain a Chow-theoretic decomposition of the diagonal:

**Proposition 2.1.** *Let  $X$  be a smooth projective  $k$ -variety of dimension  $n$ . Suppose there exists  $Z \in \mathrm{CH}^n(X \times_k X)$ , supported on  $D \times_k X$  for some proper closed subset  $D \subset X$ , and  $x \in X(k)$  such that*

$$\Delta_X - X \times x = Z \text{ in } B^n(X \times_k X).$$

*Then  $X$  admits a Chow-theoretic decomposition of the diagonal.*

*Proof.* It is proposition 2.1 of [32] since even in positive characteristic, cycles algebraically equivalent to 0 are nilpotent for the composition of self-correspondences (see [30] and [29] which use no assumptions on  $\mathrm{char}(k)$ ).  $\square$

We recall the following classical fact on a cubic hypersurface.

**Proposition 2.2.** *Let  $X$  be a smooth cubic hypersurface of dimension 3. Then  $X$  admits a degree 2 dominant rational map  $\mathbb{P}^3 \dashrightarrow X$ . It follows that  $2\Delta_X$  admits a decomposition*

$$2\Delta_X = 2(X \times_k x) + Z \text{ in } \mathrm{CH}^3(X \times_k X)$$

*where  $x \in X(k)$  and  $Z$  is supported on  $D \times_k X$  for some divisor  $D \subsetneq X$ .*

*Sketch of proof.* The first fact is classical and is presented, for example in appendix B of [4]. We recall briefly the construction of the degree 2 map from a rational variety. Let  $l_0$  be a general line in  $X$  then the map  $P(T_{X|l_0}) \dashrightarrow X$  taking a point  $(x, v)$  (with  $x \in l_0$  and  $v \in P(T_{X,x})$ ) such that the line  $\langle x, v \rangle$  is tangent to  $X$  at  $x$  but not contained in  $X$ , to the other point of the intersection  $X \cap \langle x, v \rangle$ , is generically finite and  $2 : 1$ .

So we have a rational map  $\varphi : \mathbb{P}_k^3 \dashrightarrow X$  of degree 2. Since resolution of singularities for threefolds exists in  $\mathrm{char}(k) > 0$  by work of Cossart and Piltant ([7] and [8]), there is a smooth projective  $k$ -variety  $\Gamma$ , with a birational morphism  $p : \Gamma \rightarrow \mathbb{P}_k^3$  and a degree 2 morphism  $\phi : \Gamma \rightarrow X$ , resolving the indeterminacies of  $\varphi$ . We have the following lemma:

**Lemma 2.3.** ([5, Proposition 6.3]). *Let  $f : Z \rightarrow Y$  be a birational morphism of smooth geometrically integral projective varieties over a field  $L$ . Then  $\mathrm{CH}_0^0(Z) \simeq \mathrm{CH}_0^0(Y)$ , where  $\mathrm{CH}_0^0(T)$ , for a proper  $L$ -variety  $T$ , is the group of 0-cycles of degree 0.*

Applying the lemma to the morphism obtained from  $p$  by base change  $p_{k(\mathbb{P}_k^3)} : \Gamma_{k(\mathbb{P}_k^3)} \rightarrow \mathbb{P}_{k(\mathbb{P}_k^3)}^3$  yields  $\mathrm{CH}_0(\Gamma_{k(\mathbb{P}_k^3)}) \simeq \mathbb{Z}$  i.e.  $\Gamma$  has universally trivial  $\mathrm{CH}_0$  group. Then, by base change we have the morphism  $\phi_{k(X)} : \Gamma_{k(X)} \rightarrow X_{k(X)}$ . The 0-cycle of  $X_{k(X)}$ ,  $\delta_X - k(X) \times_k x$  has degree 0, where  $\delta_X \in X(k(X))$  is the diagonal point (the image of the generic point of  $X$  by the diagonal morphism) and  $x \in X(k)$ . Since  $\phi_{k(X)}$  is a degree 2 proper morphism, we have  $\phi_{k(X),*} \phi_{k(X)}^*(\delta_X - k(X) \times_k x) = 2(\delta_X - k(X) \times_k x)$  but since that operation factors through  $\mathrm{CH}_0^0(\Gamma_{k(X)})$ , which is zero, we have  $2(\delta_X - k(X) \times_k x) = 0$ .  $\square$

For a smooth projective  $k$ -variety, the second punctual Hilbert scheme  $X^{[2]}$  of  $X$  is obtained as the quotient of the blow-up  $\widehat{X \times_k X}$  of  $X \times_k X$  along the diagonal by its natural involution. Let  $\mu : X \times_k X \dashrightarrow X^{[2]}$  be the natural rational map and  $r : \widehat{X \times_k X} \rightarrow X^{[2]}$  be the quotient morphism. We collect some results of [32] whose proofs are essentially the same. So we just mention, when needed, the change needed or the facts required in characteristic  $p > 2$ :

**Lemma 2.4.** ([32, Lemma 2.3]). *Let  $X$  be a smooth projective variety of dimension  $n$ . Then there exists a codimension  $n$  cycle  $Z$  in  $X^{[2]}$  such that  $\mu^*Z = \Delta_X$  in  $\mathrm{CH}^n(X \times_k X)$ .*

**Corollary 2.5.** ([32, Corollary 2.4]). *Any symmetric codimension  $n$  cycle on  $X \times_k X$  is rationally equivalent to  $\mu^*\Gamma$  for a codimension  $n$  cycle  $\Gamma$  on  $X^{[2]}$ .*

**Lemma 2.6.** ([32, Lemma 2.5]). *Let  $X$  be smooth projective  $k$ -variety of dimension  $n$ . Suppose  $X$  admits a  $\mathbb{Z}_2$ -cohomological decomposition of the diagonal*

$$[\Delta_X - x \times_k X] = [Z] \text{ in } H^{2n}(X \times_k X, \mathbb{Z}_2(n))$$

*where  $Z$  is a cycle supported on  $D \times_k X$  for some proper closed subset  $D$  of  $X$  and  $x$   $k$ -rational point of  $X$ . Then  $X$  admits a  $\mathbb{Z}_2$ -cohomological decomposition of the diagonal*

$$[\Delta_X - x \times_k X - X \times_k x] = [W] \text{ in } H^{2n}(X \times_k X, \mathbb{Z}_2(n)),$$

*where  $W$  is a cycle supported on  $D \times_k X$  and  $W$  is invariant under the natural involution of  $X \times_k X$ .*

The following result is proved in [32, Proposition 2.6] over  $\mathbb{C}$ .

**Proposition 2.7.** *Let  $X$  be a smooth odd degree complete intersection of odd dimension  $n$ . If  $X$  admits a  $\mathbb{Z}_2$ -cohomological decomposition of the diagonal, there exists a cycle  $\Gamma \in \mathrm{CH}^n(X^{[2]})$  with the following properties:*

- (1)  $\mu^*\Gamma = \Delta_X - x \times_k X - X \times_k x - W$  in  $\mathrm{CH}^n(X \times_k X)$ , with  $W$  supported on  $D \times_k X$ , for some closed proper subset  $D \subset X$ .
- (2)  $[\Gamma] = 0$  in  $H^{2n}(X^{[2]}, \mathbb{Z}_2(n))$ .

*Sketch of proof of 2.7.* The proof uses an analysis of the cohomology group  $H^{2n}(X^{[2]}, \mathbb{Z}_2(n))$  and more precisely of the morphism  $j_{E_X*} : H^{2n-2}(E_X, \mathbb{Z}_2) \rightarrow H^{2n}(X^{[2]}, \mathbb{Z}_2(n))$ , where  $E_X$  is the exceptional divisor of  $X^{[2]}$ . This analysis is delicate for even dimensional odd degree complete intersections, but for odd dimension and odd degree complete intersections, the restriction map from the even degree cohomology of projective space to the even degree cohomology of  $X$  is surjective, so the result follows from the analysis of the cohomology of  $(\mathbb{P}^N)^{[2]}$  which is Chow theoretic and works in any characteristic.

The only additional fact to check is that the cohomology of  $X^{[2]}$  has no 2-torsion when  $X$  is an odd degree complete intersection in projective space. To see this, choose a smooth projective lifting of  $X$  to characteristic 0 over a discrete valuation ring  $\mathfrak{X} \rightarrow \mathrm{Spec}(R)$ . As a zero dimensional length two subscheme is local complete intersection, and has trivial degree 1 coherent cohomology,  $\mathrm{Hilb}_2(\mathfrak{X}/\mathrm{Spec}(R)) \rightarrow \mathrm{Spec}(R)$  is smooth and projective by [16, Proposition I.2.15.4] (and smoothness of the fibers  $\mathfrak{X}_{\bar{\eta}}^{[2]}$  and  $X^{[2]}$ ). So,  $H^r(\mathfrak{X}_{\bar{\eta}}^{[2]}, \mathbb{Z}_2) \simeq H^r(X^{[2]}, \mathbb{Z}_2) \forall r \geq 0$  by the smooth proper base change. Now by the comparison theorem,  $H^r(\mathfrak{X}_{\bar{\eta}}^{[2]}, \mathbb{Z}_2) \simeq H_B^r(\mathfrak{X}_{\bar{\eta}}^{[2]}, \mathbb{Z}_2)$  and by [27] these last groups have no 2-torsion. The rest of the proof works just like in [32].  $\square$

For  $X$  a smooth cubic hypersurface in  $\mathbb{P}^{n+1}$ , we recall another description of  $X^{[2]}$  used in [32]. Let  $F(X)$  be the variety of lines of  $X$  and  $P = \{([l], x), x \in l, l \subset X\}$  be the universal  $\mathbb{P}^1$ -bundle over  $F(X)$  with projections  $p : P \rightarrow F(X)$  and  $q : P \rightarrow X$ , and let  $P_2 \rightarrow F(X)$  be the  $\mathbb{P}^2$ -bundle defined as the symmetric product of  $P$  over  $F(X)$ . There is a natural embedding  $P_2 \xrightarrow{i_{P_2}} X^{[2]}$  which maps each fiber of  $P_2 \rightarrow F(X)$ , that is the second symmetric product of a line in  $X$ , isomorphically onto the set of subschemes of length 2 of  $X$  contained in this line. Let  $p_X : P_X \rightarrow X$  be the projective bundle with fiber over  $x \in X$  the set of lines in  $\mathbb{P}^{n+1}$  passing through  $x$ . Note that  $P$  is naturally contained in  $P_X$ .

**Proposition 2.8.** ([32, Proposition 2.9]). *In the above situation, we have the following properties:*

(i) *The natural map  $\Phi : X^{[2]} \dashrightarrow P_X$  which to a unordered pair of points  $x, y \in X$  not contained in a common line of  $X$  associates the pair  $([l_{x,y}], z)$ , where  $l_{x,y}$  is the line in  $\mathbb{P}^{n+1}$  generated by  $x$  and  $y$ , and  $z \in X$  is the residual point of the intersection  $l_{x,y} \cap X$ , is desingularized by the blow-up of  $X^{[2]}$  along  $P_2$ .*

(ii) *The induced morphism  $\tilde{\Phi} : \widetilde{X^{[2]}} \rightarrow P_X$  identifies  $\widetilde{X^{[2]}}$  with the blow-up  $\widetilde{P_X}$  of  $P_X$  along  $P$ .*

(iii) *The exceptional divisors of the two maps  $\widetilde{X^{[2]}} \rightarrow X^{[2]}$  and  $\widetilde{P_X} \rightarrow P_X$  are identified by the isomorphism  $\tilde{\Phi}' : \widetilde{X^{[2]}} \cong \widetilde{P_X}$  of (ii).*

*Proof of Theorem 1.1.* Let  $X \subset \mathbb{P}_k^4$  be a smooth cubic threefold. By Proposition 2.2, we see that there is a nonempty open subset  $U_0 \subset X$ , such that  $(\Delta_X - X \times_k x)|_{U_0 \times_k X}$  is a 2-torsion element of  $B^3(U_0 \times_k X)$ . The subgroup of 2-torsion elements of  $B^3(U_0 \times_k X)$  is a  $\mathbb{Z}/2\mathbb{Z}$ -module and since  $\mathbb{Z}/2\mathbb{Z}$  is a quotient of the localization  $\mathbb{Z}_{(2)}$  of  $\mathbb{Z}$  in  $2\mathbb{Z}$ , a 2-torsion element of  $B^3(U_0 \times_k X)$  is 0 if and only if it is 0 in  $B^3(U_0 \times_k X) \otimes \mathbb{Z}_{(2)}$ . Since  $\mathbb{Z}_2$  is the completion of the local ring  $\mathbb{Z}_{(2)}$  along its maximal ideal,  $\mathbb{Z}_2$  is a faithfully flat  $\mathbb{Z}_{(2)}$ -module. Hence, a 2-torsion element in  $B^3(U_0 \times_k X)$  is 0 if and only if it is 0 in  $B^3(U_0 \times_k X) \otimes \mathbb{Z}_2$ . So in order to prove that  $X$  admit a Chow-theoretic decomposition of the diagonal, we only need to check that it is 0 in  $B^3(U' \times X) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ , for some open subset  $U' \subset X$ . Once we know that, we will have that  $(\Delta_X - X \times_k x)|_{U' \times_k X}$  is 0 in  $B^3(U' \times_k X)$  i.e. a decomposition of the diagonal

$$\Delta_X = X \times_k x + Z \text{ in } B^3(X \times_k X)$$

where  $Z$  is supported on  $D \times_k X$  for some proper closed subset  $D \subsetneq X$ . Applying Proposition 2.1, this will yield the Chow-theoretic decomposition of the diagonal. So we shall work with  $\mathbb{Z}_2$  coefficients and, adapting arguments of [32], show that a cohomological decomposition of the diagonal with coefficients in  $\mathbb{Z}_2$  implies that  $(\Delta_X - X \times_k x)|_{U \times_k X} = 0$  in  $B^3(U \times X) \otimes_{\mathbb{Z}} \mathbb{Z}_2$  for some nonempty open subset  $U$  of  $X$ .

Assume  $X$  admits a cohomological decomposition of the diagonal with coefficients in  $\mathbb{Z}_2$ . The assumptions of Proposition 2.7 are satisfied by  $X$ , since the cohomology of a smooth cubic hypersurface with coefficients in  $\mathbb{Z}_2$  has no torsion and is algebraic in even degree. Using the notation introduced previously, there exists, by Proposition 2.7, a cycle  $\Gamma \in \mathrm{CH}^3(X^{[2]})$  such that

$$(3) \quad \mu^* \Gamma = \Delta_X - x \times_k X - X \times_k x - W \text{ in } \mathrm{CH}^3(X \times_k X),$$

with  $W$  supported on  $D \times_k X$ ,  $D \subsetneq X$ , and  $[\Gamma] = 0$  in  $H^6(X^{[2]}, \mathbb{Z}_2(3))$ .

By Proposition 2.8, the blow-up  $\sigma : \widetilde{X^{[2]}} \rightarrow X^{[2]}$  of  $X^{[2]}$  along  $P_2$  identifies via  $\tilde{\Phi}$  with the blow-up  $\widetilde{P_X}$  of  $P_X$  along  $P$ . Furthermore, the exceptional divisor  $E \xrightarrow{i_E} \widetilde{X^{[2]}}$  of  $\tilde{\Phi} : \widetilde{X^{[2]}} \rightarrow P_X$  is also the exceptional divisor of  $\sigma : \widetilde{X^{[2]}} \rightarrow X^{[2]}$ , hence maps via  $\sigma$  to  $P_2 \subset X^{[2]}$ . Since  $\tilde{\Phi}$  is a blow-up of a smooth subvariety, the Chow groups of  $\widetilde{X^{[2]}}$  decomposes as  $\mathrm{CH}^*(\widetilde{X^{[2]}}) = \tilde{\Phi}^* \mathrm{CH}^*(P_X) \oplus i_{E*} \mathrm{CH}^*(E)$ ; we have a similar decomposition for the cohomology groups  $H^*(\widetilde{X^{[2]}}, \mathbb{Z}_2) = \tilde{\Phi}^* H^*(P_X, \mathbb{Z}_2) \oplus i_{E*} H^*(E, \mathbb{Z}_2)$  and these decompositions are compatible with the cycle map. By work of Shen [26, Theorem 1.1], the group of 1-cycles of  $X$  is generated by lines of  $X$  i.e. the action of correspondence  $P$  induces a surjective morphism

$P_* : \text{CH}_0(F(X)) \rightarrow \text{CH}_1(X)$ .

So let  $\gamma$  be a 1-cycle homologically trivial with coefficients in  $\mathbb{Z}_2$  on  $X$ ; we can write it  $P_*(z)$  for a  $z \in \text{CH}_0(F(X)) \otimes \mathbb{Z}_2$ . The degree of  $\gamma \cdot H$ , where  $H = c_1(\mathcal{O}_X(1))$ , is 0 in  $\mathbb{Z}_2$  since it is algebraically trivial in  $\text{CH}_0(X) \otimes \mathbb{Z}_2 \xrightarrow{\deg} \mathbb{Z}_2$ . But, with the above notations  $\gamma \cdot H = P_*(z) \cdot H = q_* p^*(z \cdot q^* H)$  and  $q^* H$  is the relative hyperplane divisor of the projective bundle  $p : P \rightarrow F(X)$ ; so that the degree of  $z$  is also 0 in  $\text{CH}_0(F(X)) \otimes \mathbb{Z}_2$  i.e.  $z$  is algebraically trivial. So  $\gamma = P_*(z)$  is algebraically trivial in  $\text{CH}_1(X) \otimes \mathbb{Z}_2$ . So algebraic and homological equivalences with coefficients in  $\mathbb{Z}_2$  coincide on  $\text{CH}_1(X)$ . Since these relations coincide also on  $\text{CH}_2(X) = \text{Pic}(X)$  and  $\text{CH}_0(X)$ , they coincide on the Chow ring of  $X$ .

Then, since  $P_X$  is a projective bundle over  $X$ , the two equivalence relations coincide also on  $P_X$ . On the other hand,  $F(X)$  being a surface, algebraic and homological equivalences coincide on  $F(X)$  hence also on the projective bundle  $P$  over  $F(X)$ . Since  $E$  is also a projective bundle over  $P$ , the two equivalence relations coincide on the blow-up  $\tilde{P}_X$  of  $P_X$  along  $P$ , which is isomorphic to  $\tilde{X}^{[2]}$ .

Now  $\sigma^*[\Gamma]$  is 0 in  $H^6(\tilde{X}^{[2]}, \mathbb{Z}_2(3))$  since  $[\Gamma] = 0$  so that  $\sigma^* \Gamma = 0$  in  $B^3(\tilde{X}^{[2]}) \otimes \mathbb{Z}_2$ . We conclude that  $\Gamma = 0$  in  $B^3(X^{[2]}) \otimes \mathbb{Z}_2$  since  $\sigma_* \sigma^* = \text{id}_{\text{CH}^*(X^{[2]})}$ . So (3) yields

$$\Delta_X = x \times_k X + X \times_k x - W' \text{ in } B^3(X \times_k X) \otimes \mathbb{Z}_2$$

where  $W'$  is supported on  $D' \times_k X$  for some  $D' \subsetneq X$ . So  $(\Delta_X - X \times_k x)|_{U \times_k X} = 0$  in  $B^3(U \times_k X) \otimes \mathbb{Z}_2$  (with  $U = X \setminus D'$ ) as we wanted.  $\square$

### 3. COHOMOLOGICAL DECOMPOSITION OF THE DIAGONAL

We prove in this section Theorem 1.3, again adapting arguments of Voisin presented in [32]. We begin by the following theorem which was proved in [32, Theorem 3.1] over  $\mathbb{C}$ .

**Theorem 3.1.** *Let  $X$  be a smooth projective  $k$ -variety of dimension  $n > 0$  and  $\ell \neq p$  a prime number.*

- (1) *Assume  $H^*(X, \mathbb{Z}_\ell)$  has no torsion,  $H^{2i}(X, \mathbb{Z}_\ell(i))$  is algebraic for  $2i \neq n$ ,  $H^{2i+1}(X, \mathbb{Z}_\ell) = 0$  for  $2i+1 \neq n$  and that  $X$  satisfies the following condition:*

(\*) *There exist finitely many smooth projective varieties  $Z_i$  of dimension  $n-2$ , correspondences  $\Gamma_i \in \text{CH}^{n-1}(Z_i \times_k X)$ , and  $n_i \in \mathbb{Z}_2$ , such that for any  $\alpha, \beta \in H^n(X, \mathbb{Z}_\ell)$ ,*

$$(4) \quad \langle \alpha, \beta \rangle_X = \sum_i n_i \langle \Gamma_i^* \alpha, \Gamma_i^* \beta \rangle_{Z_i}.$$

*Then  $X$  admits a cohomological decomposition of the diagonal with coefficients in  $\mathbb{Z}_\ell$ .*

- (2) *If  $n = 3$ ,  $\text{char}(k) \neq \ell$  and  $X$  admits a cohomological decomposition of the diagonal with coefficients in  $\mathbb{Z}_\ell$ , then (\*) is satisfied.*

*Sketch of proof.* The adaptation of the proof given in [32], to positive characteristic is straightforward; the only fact to use for the second point is the existence of (embedded) resolution of singularities in dimension 3 for algebraically closed fields of positive characteristic (see [7] and [8]).  $\square$

#### 3.1. The intermediate Jacobian of a cubic threefold and Abel-morphisms.

Let us denote  $\text{CH}_{\text{alg}}^2(Y)$  the group of codimension 2 cycles algebraically equivalent to zero on a  $k$ -variety  $Y$ . Given an abelian variety  $Ab$  over  $k$ , following [20, VIa], we shall call a (group) homomorphism  $f : \text{CH}_{\text{alg}}^2(Y) \rightarrow Ab$  a regular morphism if for any smooth quasi-projective  $k$ -variety  $T$  and  $Z \in \text{CH}^2(T \times_k Y)$  such that for any  $t \in T(k)$ ,  $Z_t \in \text{CH}_{\text{alg}}^2(Y)$ , the composition  $T \rightarrow \text{CH}_{\text{alg}}^2(Y) \xrightarrow{f} Ab$  is a morphism of algebraic varieties. We say that  $\text{CH}_{\text{alg}}^2(Y)$  admits an algebraic representative if there is an abelian variety  $Ab(Y)$  over  $k$  and a regular morphism  $\phi : \text{CH}_{\text{alg}}^2(Y) \rightarrow Ab(Y)$  which is universal in the sense that any regular

morphism factor as a composition of  $\phi$  followed by a morphism of algebraic varieties. In that case we call the morphism  $\phi_Z : T \rightarrow \mathrm{CH}_{\mathrm{alg}}^2(Y) \xrightarrow{f_Y} \mathrm{Ab}(Y)$  the Abel-Jacobi morphism induced by  $Z$ . By [22, Theorem 1.9],  $\mathrm{CH}_{\mathrm{alg}}^2(Y)$  admits an algebraic representative when  $Y$  is a smooth projective variety over an algebraically closed field.

Now, let  $X \subset \mathbb{P}_k^4$  be a smooth cubic hypersurface. The linear projection  $\mathbb{P}_k^4 \dashrightarrow \mathbb{P}_k^2$  centered along a general line  $l \subset X$  (see for example [21, Proposition 1.25]) gives a rational map  $X \dashrightarrow \mathbb{P}_k^2$  which after blowing up  $l$ , yields an ordinary conic bundle  $\tilde{X} \rightarrow \mathbb{P}_k^2$ . By results of Beauville ([3, Theorem 3.1 and Proposition 3.3]), the Prym variety  $A$  associated to the conic bundle is the algebraic representative of  $\mathrm{CH}_{\mathrm{alg}}^2(\tilde{X}) \simeq \mathrm{CH}_{\mathrm{alg}}^2(X)$  and  $\mathrm{CH}_{\mathrm{alg}}^2(X) = A(k)$ . The principally polarized abelian variety  $A$  obtained by this construction is independent of the choice of a general  $l$ . So we call  $J(X) := A$  the intermediate Jacobian of  $X$ ; it is a 5-dimensional abelian variety endowed with the principal polarization  $\theta$  of  $A$ . By results of Beauville ([3, Remark 2.7]), we know that there is an isomorphism of  $\mathbb{Z}_2$ -modules with their intersection forms

$$(5) \quad t : (H^1(J(X), \mathbb{Z}_2), \theta) \rightarrow (H^3(\tilde{X}, \mathbb{Z}_2), \langle \cdot, \cdot \rangle_{\tilde{X}}) \simeq (H^3(X, \mathbb{Z}_2), \langle \cdot, \cdot \rangle_X)$$

It is known (see Murre [23, section VI], see also [4] over  $\mathbb{C}$ ) that the Abel-Jacobi morphism associated to the universal  $\mathbb{P}^1$ -bundle  $P \subset F(X) \times X$  over the variety of lines  $F(X)$  of  $X$ , induces an isomorphism of abelian varieties  $\phi_P : \mathrm{Alb}(F(X)) \simeq J(X)$  where  $\mathrm{Alb}(F(X))$  is the Albanese variety of  $F(X)$ , which is defined in this setting as the dual of the Picard variety  $\mathrm{Pic}^0(F(X))$ .

Since  $F(X)$  is the zero locus of a regular section of the vector bundle  $\mathcal{E} = \mathrm{Sym}_3(E)$  on the grassmannian  $Gr(2, 5)$ , where  $E$  is the rank 2 quotient bundle on  $Gr(2, 5)$ , we have the following exact sequence:

$$0 \rightarrow \wedge^4 \mathcal{E}^* \rightarrow \wedge^3 \mathcal{E}^* \rightarrow \wedge^2 \mathcal{E}^* \rightarrow \mathcal{E}^* \rightarrow \mathcal{O}_{Gr(2,5)} \rightarrow \mathcal{O}_{F(X)} \rightarrow 0$$

given by the Koszul resolution of the sheaf of ideals of  $F(X)$  in  $Gr(2, 5)$  (the exactness follows from the fact that  $F(X)$  has the codimension  $4 = \mathrm{rank}(\mathcal{E})$ ) so, we have a quasi-isomorphism of complexes  $\wedge \mathcal{E}^* \simeq \mathcal{O}_{F(X)}[4]$ .

Now, we have a spectral sequence  $E_1^{p,q} = H^q(Gr(2, 5), \wedge^{4-p} \mathcal{E}^*) \Rightarrow H^{p+q-4}(F(X), \mathcal{O}_{F(X)})$ , which, according to [1, Theorem 5.1], degenerates at  $E_1$ , so that  $H^1(F(X), \mathcal{O}_{F(X)}) \simeq H^3(Gr(2, 5), \wedge^2 \mathcal{E}^*)$ . According to [1, Proposition 5.11 and Lemma 5.7], we have an isomorphism  $H^3(Gr(2, 5), \wedge^2 \mathcal{E}^*) \simeq H^0(\mathbb{P}_k^4, T\mathbb{P}_k^4(-1))$ . By the Euler sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_k^4}(-1) \rightarrow H^0(\mathbb{P}_k^4, \mathcal{O}_{\mathbb{P}_k^4}(1))^\vee \otimes \mathcal{O}_{\mathbb{P}_k^4} \rightarrow T\mathbb{P}_k^4 \rightarrow 0,$$

we have  $H^1(F(X), \mathcal{O}_{F(X)}) \simeq H^0(\mathbb{P}_k^4, \mathcal{O}_{\mathbb{P}_k^4}(1))^\vee$ . Tensoring the normal bundle exact sequence of the inclusion  $X \subset \mathbb{P}_k^4$  with  $\Omega_{X/k}^3 \simeq \mathcal{O}_{\mathbb{P}_k^4}(-2)|_X$  and looking at the associated long exact sequence, we have  $H^0(X, \mathcal{O}_X(1)) \simeq H^1(X, \Omega_{X/k}^2)$ . Since  $H^0(X, \mathcal{O}_X(1)) \simeq H^0(\mathbb{P}_k^4, \mathcal{O}_{\mathbb{P}_k^4}(1))$ , we have  $H^1(F(X), \mathcal{O}_{F(X)}) \simeq H^1(X, \Omega_{X/k}^2)^\vee$ . Since by [23, Theorem 8], the Picard and Albanese varieties of  $F(X)$  are isomorphic, we have  $T_0 J(X) \simeq H^2(X, \Omega_{X/k})$ .

In order to apply Voisin's method in [31] to analyse the existence of a decomposition of the diagonal for a cubic threefold, we need to make sure there exists, as it is the case in characteristic 0 ([17] and [15]), a parametrization of  $J(X)$  with separably rationally connected generic fiber, namely condition (\*) of the introduction. This parametrization will be constructed in Section 4. So let us proceed to the proof of the criterion (Theorem 1.3 of the introduction) for the existence of a cohomological decomposition of the diagonal assuming that (\*) holds.

**3.2. Decomposition of the diagonal for a smooth cubic threefold.** In this section, we prove the following theorem which was first proved over  $\mathbb{C}$  in [31, Theorem 4.9], [32, Theorem 4.1]. Item (ii) is specific to the finite field situation and is Theorem 1.4 of the introduction.

**Theorem 3.2.** (i) Let  $X \subset \mathbb{P}_k^4$  be a smooth cubic hypersurface ( $k = \bar{k}$  and  $\text{char}(k) > 2$ ). Then  $X$  admits a cohomological (hence Chow-theoretic by Theorem 1.1) decomposition of the diagonal if and only if there is a  $\gamma \in \text{CH}_1(J(X)) \otimes_{\mathbb{Z}} \mathbb{Z}_2$  such that  $\theta^4/4! = [\gamma]$  in  $H^8(J(X), \mathbb{Z}_2(4))$ .

(ii) If  $k = \overline{\mathbb{F}_p}$ ,  $p > 2$  and the Tate conjecture is true for divisors on surfaces over finite fields, then every smooth cubic hypersurface of  $\mathbb{P}_k^4$  admits a Chow-theoretic decomposition of the diagonal.

*Proof.* Assume  $\theta^4/4! \in \text{CH}_1(J(X)) \otimes \mathbb{Z}_2$ . We will prove that  $X$  admits a cohomological (with coefficient in  $\mathbb{Z}_2$ ) decomposition of the diagonal. We know that  $H^*(X, \mathbb{Z}_2)$  has no torsion so applying Künneth decomposition, we can write  $[\Delta_X] = \sum_{i=0}^6 \delta_{i,6-i}$  where  $\delta_{i,6-i} \in H^i(X, \mathbb{Z}_2) \otimes H^{6-i}(X, \mathbb{Z}_2)$  are the components of  $[\Delta_X] \in H^6(X \times_k X, \mathbb{Z}_2)$ . Since  $H^1(X, \mathbb{Z}_2) = 0 = H^5(X, \mathbb{Z}_2)$  we have  $\delta_{1,5} = 0 = \delta_{5,1}$ . We know that  $\delta_{6,0}$  is the class  $X \times_k x$  for any point  $x \in X(k)$  and  $\delta_{0,6}$  is the class of the subvariety  $x \times_k X$  which obviously does not dominate  $X$  by the first projection. Since  $H^2(X, \mathbb{Z}_2)$  and  $H^4(X, \mathbb{Z}_2)$  are algebraic,  $\delta_{2,4}$  and  $\delta_{4,2}$  are linear combinations of classes of algebraic subvarieties of  $X \times_k X$  that do not dominate  $X$  by the first projection. The existence of a cohomological decomposition of the diagonal with coefficients in  $\mathbb{Z}_2$  is thus equivalent to the existence a cycle  $Z \subset X \times_k X$  such that the support of  $Z$  is contained in  $D \times_k X$ , with  $D \subset X$  a proper subscheme, and  $Z^* : H^3(X, \mathbb{Z}_2) \rightarrow H^3(X, \mathbb{Z}_2)$  is the identity map since in this case  $\delta_{3,3} = [Z]$ . We proceed as in [31, Theorem 4.9] to construct such a  $Z$ .

Let  $C = \sum_i m_i C_i$  be a 1-cycle of  $J(X)$  of class  $\theta^4/4!$  in  $H^8(J(X), \mathbb{Z}_2)$ , where  $C_i \subset J(X)$  are curves and  $m_i \in \mathbb{Z}_2$ . According to condition (\*), which is Theorem 4.7, there are a smooth 10-dimensional quasi-projective  $k$ -variety  $B$  and a cycle  $\mathcal{Z} \subset B \times_k X$ , flat over  $B$ , such that the induced Abel-Jacobi morphism  $\phi_{\mathcal{Z}} : B \rightarrow J(X)$  is dominant with general fiber  $\mathbb{P}_k^5$ . Let  $B'$  be the closure of the graph of  $\phi_{\mathcal{Z}}$  in  $\overline{B} \times J(X)$ , where  $\overline{B}$  is a compactification of  $B$ ;  $B'$  is birational to the quasi-projective variety  $B$  and the projection yields a proper, surjective morphism  $\phi : B' \rightarrow J(X)$  with general fiber  $\mathbb{P}_k^5$ . Let  $W \subset J(X)$  be an open subscheme contained in the image of  $\phi_{\mathcal{Z}}$  such that  $\forall x \in W(k)$ ,  $\phi^{-1}(x) \simeq \mathbb{P}_k^5$ . Using Chow's moving lemma, we can assume that the generic point of each  $C_i$  is in  $W$ . Let  $n_i : \widetilde{C}_i \rightarrow C_i$  be the normalization of  $C_i$ . Let  $N_i : B_i \rightarrow B' \times_{C_i} \widetilde{C}_i$  be the normalization morphism. A component of  $B_i$  is proper and flat (because integral over a smooth curve) over the smooth curve  $\widetilde{C}_i$  with general fiber  $\mathbb{P}_k^5$  so by Tsen's Theorem ([28]),  $B_i \rightarrow \widetilde{C}_i$  admits a section  $\sigma_i$  and  $\phi \circ n'_i \circ N_i \circ \sigma_i = n_i$  (where  $n'_i : B' \times_{C_i} \widetilde{C}_i \rightarrow B'$  is the projection). Let  $Z_i \subset \widetilde{C}_i \times_k X$  be the cycle  $(pr_B^{B \times_k J(X)} \circ n'_i \circ N_i \circ \sigma_i, id_X)^* \mathcal{Z}$ . We have the easy equality:

**Lemma 3.3.** The homomorphisms  $Z_{i,*}$  and  $t \circ \phi_{Z_i,*} : H^1(\widetilde{C}_i, \mathbb{Z}_2) \rightarrow H^3(X, \mathbb{Z}_2)$  coincide (where  $t : (H^1(J(X), \mathbb{Z}_2), \theta) \rightarrow (H^3(X, \mathbb{Z}_2), \langle \cdot, \cdot \rangle_X)$  is the isomorphism).

Let  $Z \subset X \times_k X$  be the cycle  $\sum_i m_i Z_i \circ {}^t Z_i$ . We have

$$(Z_i \circ {}^t Z_i)^* = Z_{i,*} \circ Z_i^* = (t^{-1})^* \circ \phi_{Z_i,*} \circ \phi_{Z_i}^* \circ t^{-1}$$

where  $(t^{-1})^*$  is the Poincaré dual of  $t^{-1}$ . But  $\phi_{Z_i}$  is just  $n_i$  so  $\phi_{Z_i,*} \circ \phi_{Z_i}^*$  is  $n_{i,*} \circ n_i^*$  which is just  $[C_i] \cup : H^1(J(X), \mathbb{Z}_2) \rightarrow H^9(J(X), \mathbb{Z}_2)$ . Hence  $Z^*$  is the composite map

$$H^3(X, \mathbb{Z}_2) \xrightarrow{t^{-1}} H^1(J(X), \mathbb{Z}_2) \xrightarrow{(\theta^4/4!) \cup = \sum_i [C_i] \cup} H^9(J(X), \mathbb{Z}_2) \xrightarrow{(t^{-1})^*} H^3(X, \mathbb{Z}_2).$$

So  $Z^*$  is the identity on  $H^3(X, \mathbb{Z}_2)$ . On the other hand,  $Z$  does not dominate  $X$  by the first projection since it is supported on  $\bigcup_i \Sigma_i \times \Sigma_i$ , where  $\Sigma_i$  is the image in  $X$  of  $\text{Supp}(Z_i)$ .

We prove now the second direction. Assume  $X$  admits a Chow-theoretic decomposition of the diagonal. Then  $X$  admits a cohomological decomposition of the diagonal with coefficients in  $\mathbb{Z}_2$  so by Theorem 3.1 we have finitely many smooth projective curves  $Z_i$  and for each curve, a correspondence  $\Gamma_i \in \text{CH}^2(Z_i \times_k X)$  and a  $n_i \in \mathbb{Z}_2$  satisfying (4). The Abel-Jacobi map  $\Phi_X$  of  $X$  induces (after choosing a reference point in  $Z_i$ ) a morphism

$$\gamma_i = \Phi_X \circ \Gamma_{i*} : Z_i \rightarrow J(X)$$



with image  $Z'_i := \gamma_{i*} Z_i \in \text{CH}_1(J(X))$ . Now we have  $\wedge^2 H^1(J(X), \mathbb{Z}_2) \simeq H^2(J(X), \mathbb{Z}_2) = H^8(J(X), \mathbb{Z}_2)^*$  and an isomorphism  $(H^1(J(X), \mathbb{Z}_2), \theta) \xrightarrow{t} (H^3(X, \mathbb{Z}_2), \langle, \rangle_X)$  with their intersection form (5). For all  $\alpha \in H^1(J(X), \mathbb{Z}_2)$ ,  $\gamma_i^* \alpha = \Gamma_i^* t(\alpha)$  so for all  $\alpha, \beta \in H^1(J(X), \mathbb{Z}_2)$ ,

$$\begin{aligned} \langle \sum_i n_i [Z'_i], \alpha \cup \beta \rangle_{J(X)} &= \sum_i n_i \langle [Z'_i] \cup \alpha, \beta \rangle_{J(X)} \\ &= \sum_i n_i \langle \gamma_i^* \alpha, \gamma_i^* \beta \rangle_{Z_i} \\ &= \sum_i n_i \langle \Gamma_i^* t(\alpha), \Gamma_i^* t(\beta) \rangle_{Z_i} \\ &= \langle t(\alpha), t(\beta) \rangle_X \text{ (by 4)} \\ &= \theta(\alpha, \beta) \\ &= \langle \frac{\theta^4}{4!}, \alpha \cup \beta \rangle_{J(X)} \end{aligned}$$

hence  $\frac{\theta^4}{4!} = \sum_i n_i [Z'_i]$ .

(ii) If the Tate conjecture is true for divisors on surfaces defined over finite fields, then the theorem of Schoen ([25]) says that the cycle map  $\text{CH}_1(J(X)) \otimes \mathbb{Z}_2 \rightarrow \bigcup_U H^8(J(X), \mathbb{Z}_2(4))^U$ , where  $U$  runs through all open subgroups of  $\text{Gal}(k/k_{\text{def}})$  ( $k_{\text{def}}$  being a finite field over which  $J(X)$  is defined), is surjective. Since  $\theta^4$  is algebraic,  $\theta^4/4! \in \bigcup_U H^8(J(X), \mathbb{Z}_2(4))^U$  and we conclude by point (i) of the theorem.  $\square$

#### 4. PARAMETRIZATION OF THE INTERMEDIATE JACOBIAN OF A SMOOTH CUBIC THREEFOLD

The goal of this section is to prove that condition (\*) still hold in the positive characteristic setting. Over  $\mathbb{C}$ , such a parametrization was achieved using the space of smooth normal elliptic quintics which we do not know to exist a priori in our setting. So we will construct some stable normal elliptic quintics using the lines on the cubic threefold.

**4.1. Some facts on the Fano variety of lines.** Let  $X$  be a smooth cubic hypersurface  $\mathbb{P}_k^4$ . Since  $\mathbb{P}_k^3$  is separably rationally connected and there is a dominant degree 2 rational map  $\mathbb{P}_k^3 \dashrightarrow X$  and  $2 \neq \text{char}(k)$ ,  $X$  is separably rationally connected.

The Fano variety of lines  $F(X) = \{[l] \in \text{Gr}(2, 5), l \subset X\}$  of  $X$  is a smooth projective surface. Denote by  $P \xrightarrow{p} F(X)$  the universal  $\mathbb{P}^1$ -bundle and by  $q : P \rightarrow X$  the projection on  $X$ .

We collect some results from Murre ([21]) on the geometry of  $F(X)$ . Let  $\mathcal{F}_0 \subset F(X)$  be the subset defined by

$$\mathcal{F}_0 = \{[l] \in F(X), \exists K \subset \mathbb{P}_k^4 \text{ a 2-plane s.t. } K \cap X = 2l + l' \text{ as divisors in } K\}.$$

Then  $\mathcal{F}_0$  is a non-singular curve ([21, Corollary 1.9]), the lines  $[l] \in \mathcal{F}_0$  are said to be of the second type and those not in  $\mathcal{F}_0$  are said to be of the first type. The subscheme

$$\mathcal{F}'_0 = \{[l] \in F(X), \exists K \subset \mathbb{P}_k^4 \text{ a 2-plane and } l' \in F(X) \text{ s.t. } K \cap X = l + 2l'\}$$

has dimension at most 1 ([21, Lemma 1.11]). For  $[l] \in F(X)$ , let us denote by

$$\mathcal{H}(l) = \overline{\{[l'] \in F(X), [l] \neq [l'], l \cap l' \neq \emptyset\}}.$$

Then  $\mathcal{H}(l)$  is a curve in  $F(X)$ . We have the following properties:

**Proposition 4.1.** ([21, (1.17), (1.18), (1.24), (1.25)]) (i) If  $[l]$  is a line of the first type on  $X$  and  $x \in l$  then there are only finitely many (in fact at most 6) lines on  $X$  through  $x$ . Moreover there is no 2-plane tangent to  $X$  in all points of  $l$ .

(ii) There is an nonempty open subscheme  $\mathcal{U}' \subset F(X)$  contained in  $F(X) \setminus (\mathcal{F}_0 \cup \mathcal{F}'_0)$  such that any  $[l] \in \mathcal{U}'$  ( $l$  is of the first type) is contained in a smooth hyperplane section of  $X$  and  $\mathcal{H}(l)$  is a smooth irreducible curve of genus 11.

By the jacobian criterion, given a hyperplane  $H \subset \mathbb{P}_k^4$ ,  $X \cap H$  is not a smooth cubic surface if and only if there is a  $x \in X$  such that  $H$  is tangent to  $X$  at  $x$ . Looking at the Gauss map  $\mathcal{D} : X \rightarrow (\mathbb{P}_k^4)^*$ , ([4]) we see that the variety  $\mathcal{D}(X) \subset (\mathbb{P}_k^4)^*$  parametrizing those hyperplanes tangent to  $X$  at some point  $x \in X$  is a hypersurface in  $(\mathbb{P}_k^4)^*$ . So the general (parametrized by the open subscheme  $(\mathbb{P}_k^4)^* \setminus \mathcal{D}(X)$ ) hyperplane section of  $X$  is smooth. Moreover, if  $[l] \in \mathcal{U}'$ , we know that  $K_l^* = \{[H] \in (\mathbb{P}_k^4)^*, l \subset H\}$ , which is a 2-plane in  $(\mathbb{P}_k^4)^*$ ,

is not contained in  $\mathcal{D}(X)$ , hence  $\mathcal{D}(X) \cap K_l^*$  is of dimension at most 1 and  $K_l^* \setminus (\mathcal{D}(X) \cap K_l^*)$  is a 2-dimensional open subscheme. So we see that in fact, for  $[l] \in \mathcal{U}'$ , the general hyperplane containing  $l$  gives a smooth hyperplane section of  $X$ . We have also the following properties:

**Proposition 4.2.** (i) For  $[l] \in \mathcal{U}'$ ,  $\text{Im}(\mathcal{H}(l)) (= \cup_{[l'] \in \mathcal{H}(l)} l' = q(p^{-1}(\mathcal{H}(l))))$  is not contained in a fixed 2-plane. So if  $[l'] \in F(X)$  is a line distinct from  $[l]$  such that  $l \cap l' \neq \emptyset$ ,  $\mathcal{H}(l)$  and  $\mathcal{H}(l')$  have no common component.

(ii) Let  $h_{l_0} : F(X) \setminus (\mathcal{H}(l_0) \cup \{[l_0]\}) \rightarrow K_{l_0}^*$  be the morphism defined by  $[l] \mapsto [\text{span}(l, l_0)]$  for  $[l_0] \in \mathcal{U}'$ . Then  $h_{l_0}$  is dominant and there is an open subscheme  $V_{l_0} \subset \mathcal{U}' \setminus (\mathcal{H}(l_0) \cup \{[l_0]\})$  such that for  $[l] \in V_{l_0}$ ,  $h_{l_0}([l])$  gives a smooth hyperplane section.

(iii) For  $[l] \in \mathcal{U}'$ , there is an open subscheme  $\mathcal{U}'_{[l]} \subset \mathcal{U}'$  such that for all  $[l'] \in \mathcal{U}'_{[l]}$ ,  $\mathcal{H}(l') \cap \mathcal{H}(l)$  is finite. So there is an open subscheme  $\mathcal{U} \subset \mathcal{U}'$  such that  $\forall [l] \in \mathcal{U}$ ,  $\mathcal{H}(l) \cap \mathcal{U}$  is a non empty open subscheme of  $\mathcal{H}(l)$ .

*Proof.* (i) Suppose there exists  $K \subset \mathbb{P}_k^4$  a 2-plane such that  $q(p^{-1}(\mathcal{H}(l))) \subset K$ .  $p^{-1}(\mathcal{H}(l))$  is a smooth irreducible ruled surface over a curve of genus 11. Since  $X$  is smooth, it cannot contain a 2-plane, so  $q(p^{-1}(\mathcal{H}(l)))$  is contracted into a curve or a set of points in  $K \cap X$ . Since the ruled surface is irreducible,  $q(p^{-1}(\mathcal{H}(l)))$  is an irreducible closed ( $q$  proper) subscheme of  $K$  and we have  $l \subset q(p^{-1}(\mathcal{H}(l)))$  so  $l = q(p^{-1}(\mathcal{H}(l)))$ . But, by 4.1 (i), since  $l$  is of the first type, the fiber of  $q$  over a point of  $l$  is 0-dimensional. So such  $K$  does not exist.

Moreover, suppose  $[l'] \in F(X) \setminus \{[l]\}$  is such that  $l' \cap l \neq \emptyset$  and  $\mathcal{H}(l)$  (which is irreducible) is one of the components of  $\mathcal{H}(l')$ , then all (except maybe the other 4 ones passing through  $l \cap l'$ ) lines that meet  $l$  are contained in the 2-plane  $\text{span}(l, l')$ , which is impossible.

(ii) Since  $[l_0] \in \mathcal{U}'$ , the general member of  $K_{l_0}^*$  gives a smooth hyperplane section which contains some lines that do not meet  $l_0$  (i.e in  $F(X) \setminus (\mathcal{H}(l_0) \cup \{[l_0]\})$ ),  $h_{l_0}$  is dominant and the general fiber is 0-dimensional of cardinal  $< 27$ . So  $h_{l_0}|_{\mathcal{U}' \setminus \mathcal{H}(l_0)}$  is still dominant. Let  $\mathcal{C} = K_{l_0}^* \cap \mathcal{D}(X)$  be the closed subscheme of dimension  $\leq 1$  parametrizing singular hyperplane sections containing  $l_0$ . Since  $h_{l_0}$  is dominant, the closed subscheme  $h^{-1}(\mathcal{C})$  has dimension  $\leq 1$ . So the property is proved with  $V_{l_0} = \mathcal{U}' \setminus (\mathcal{H}(l_0) \cup h^{-1}(\mathcal{C}))$ .

(iii) For  $[l_0] \in \mathcal{U}'$ , according to the previous point, there is an open subscheme  $V_{l_0} \subset \mathcal{U}' \setminus (\mathcal{H}(l_0) \cup \{[l_0]\})$  such that  $\forall [l] \in V_{l_0}$ ,  $h_{l_0}([l])$  is a smooth hyperplane section, in particular,  $\forall [l] \in V_{l_0}$ ,  $\mathcal{H}(l_0) \neq \mathcal{H}(l)$  (otherwise, all these lines would be contained in the smooth cubic surface and  $k$  being algebraically closed,  $\mathcal{H}(l_0)(k)$  is infinite). For the last statement, should they exist, take finitely many  $[l_j] \in \mathcal{U}'$  such that the  $\mathcal{H}(l_j)$  are the irreducible components of the divisors  $(\mathcal{F}_0, \mathcal{F}'_0, \dots)$  removed from  $F(X)$  to obtain  $\mathcal{U}'$  and set  $\mathcal{U} = \cap_j V_{l_j}$  (and  $\mathcal{U}' = \mathcal{U}$  if there is no such  $l_j$ ).  $\square$

**4.2. Space of normal elliptic quintics.** We will construct in this section, singular normal elliptic quintic curves, namely cycles of rational curves (generically) with four components, 3 lines and a conic. We will use for this the properties of the Fano surface from 4.1 and 4.2.

Take  $[l_0] \in \mathcal{U}'$ . Let  $\mathcal{C} \subset K_{l_0}^*$  the closed subscheme of dimension  $\leq 1$  parametrizing the singular hyperplane sections containing  $l_0$ . Its preimage  $h_{l_0}^{-1}(\mathcal{C})$  has dimension at most 1 since there is at least one (hence a 2-dimensional open set of) smooth hyperplane section containing  $l_0$ . Let us denote by  $(C_i)_{1 \leq i \leq m}$  the irreducible components of  $h_{l_0}^{-1}(\mathcal{C})$  and should they exist such, let us denote by  $([l^i])_{1 \leq i \leq m} \in \mathcal{U}^m$  some lines such that  $\mathcal{H}(l^i) \cap C_i$  has dimension 1. We can choose  $[l_1]$  in the (2-dimensional) open subscheme  $\cap_{i=1}^m \mathcal{U}'_{[l^i]} \cap V_{l_0} \cap \mathcal{U}$ . Then  $\mathcal{H}(l_1) \cap h_{l_0}^{-1}(\mathcal{C}) = \cup_{i=1}^m \mathcal{H}(l_1) \cap C_i \subset \cup_{i=1}^m (\mathcal{H}(l_1) \cap \mathcal{H}(l^i))$ , the last set is finite as finite union of finite sets  $([l_1] \in \cap_{i=1}^m \mathcal{U}'_{[l^i]})$ . Since the hyperplane section  $S = H_1 \cap X$  is smooth (where  $H_1 = \text{span}(l_0, l_1)$ ) and the lines meeting  $l_0$  and  $l_1$  are in  $S$ ,  $\mathcal{H}(l_1) \cap \mathcal{H}(l_0)$  is finite.

Since  $(\mathcal{H}(l_1) \cap \mathcal{U}') \setminus (h_{l_0}^{-1}(\mathcal{C}) \cup \mathcal{H}(l_0))$  is a nonempty open subscheme of  $\mathcal{H}(l_1)$  and  $S$  contains 27 lines, we can choose  $[l_2]$  in this open subset such that  $[l_2]$  is not contained in  $S$  (in particular not contained in  $H_1$ ) and  $H_2 \cap X$  is a smooth cubic surface, where  $H_2 = \text{span}(l_0, l_2)$  and  $H_2 \neq H_1$  ( $l_2$  is not in  $S$ ). We have  $\mathcal{H}(l_2) \neq \mathcal{H}(l_0)$  (because  $[l_1] \in \mathcal{H}(l_2)$

and is not in  $\mathcal{H}(l_0)$  and  $\mathcal{H}(l_2) \neq \mathcal{H}(l_1)$  by Proposition 4.2 (i) so that  $\mathcal{H}(l_2) \setminus (\mathcal{H}(l_0) \cup \mathcal{H}(l_1))$  is a nonempty open subscheme of  $\mathcal{H}(l_2)$ . Since  $H_2 \cap X$  contains only finitely many lines, we can take  $[l_3] \in \mathcal{H}(l_2) \setminus (\mathcal{H}(l_0) \cup \mathcal{H}(l_1))$  not in that surface and not intersecting  $l_2$  at  $l_1 \cap l_2$  (there are at most 4 other lines passing through  $l_1 \cap l_2$  by Proposition 4.1 (i)).

Letting  $H_3 = \mathrm{span}(l_0, l_3)$ , we have  $H_1 \cap H_2 \not\subset H_3$  otherwise the point  $l_2 \cap l_1$  would be in  $H_1 \cap H_2 \subset H_3$  and the same with the point  $l_2 \cap l_3$  so that  $(l_2 \cap l_1 \neq l_2 \cap l_3)$  we would have  $l_2 \subset H_3$ , i.e.  $H_3 = H_2$  but  $l_3 \not\subset H_2$ . So  $H_1 \cap H_2 \cap H_3$  is the line  $l_0$ .

**Lemma 4.3.** *A hyperplane section of a smooth hypersurface of degree  $\geq 2$  in projective space  $\mathbb{P}^n$  has a zero-dimensional singular set. Hence a hyperplane section of a smooth cubic surface does not contain a double line.*

Indeed, the singular locus of a hyperplane section  $H \cap X$  of  $X$  is the fiber over the point  $[H] \in (\mathbb{P}^n)^*$  of the Gauss map of  $X$  which is given by a base point free ample linear system.

Applying the lemma, we see that the residual conic  $D$  to  $l_0$  in  $H_3 \cap S$  is the union of two (secant) distinct lines or a nondegenerate conic. We have  $D \cap l_2 = \emptyset$  otherwise, a point  $x \in D \cap l_2$  would be in  $H_1 \cap H_2 \cap H_3$  which is known to be  $l_0$  and  $l_0 \cap l_2 = \emptyset$ . As the intersection of the 2-plane  $H_1 \cap H_3$  with the line  $l_1$  in the 3-dimensional projective space  $H_1$ ,  $(H_1 \cap H_3) \cap l_1$  is a complete intersection point of  $(H_1 \cap H_3) \cap X$  not on  $l_0$  (since  $l_0 \cap l_1 = \emptyset$ ) so  $D \cap l_1$  is that point. In particular if  $D$  is degenerate, it is not the meeting point of the two components of  $D$ . The same is true for  $D \cap l_3$  which the complete intersection point  $(H_1 \cap H_3) \cap l_3$ , intersection of the 2-plane  $H_1 \cap H_3$  with the line  $l_3$  in the 3-dimensional projective space  $H_3$ . So  $C = l_1 \cup l_2 \cup l_3 \cup D$  is a locally complete intersection closed subset of  $X$  of pure dimension 1. It is a curve of arithmetic genus 1, with trivial dualizing sheaf, not contained in any hyperplane and whose intersection with a general hyperplane is a degree 5 effective zero cycle. It is thus a singular linearly normal elliptic quintic curve. From this construction, we see that we have at least a 6-dimensional family of such curves:  $[l_0], [l_1]$  are chosen in open subsets of  $F(X)$ ,  $[l_2]$  is chosen in an open subset of the curve  $\mathcal{H}(l_1)$  and  $[l_3]$  in an open subset of the curve  $\mathcal{H}(l_2)$ .

The curve  $C$  thus constructed have the following properties:

**Proposition 4.4.** (i)  $h^i(X, \mathcal{I}_C) = 0$   $i \in \{0, 1, 3\}$  and  $h^2(X, \mathcal{I}_C) = 1$ ;  
(ii)  $h^i(X, \mathcal{I}_C(1)) = 0$  for all  $0 \leq i \leq 2$ ;  
(iii)  $h^0(X, \mathcal{I}_C(2)) = 5$ ,  $h^i(X, \mathcal{I}_C(2)) = 0$  for  $i = 1, 2$ ;  
(iv)  $\dim_k \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-2)) = 1$  furthermore the generator of  $\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-2))$  generates  $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2))$  at any point of  $C$ .

*Proof.* (i) It follows immediately from the long exact sequence associated to the short exact sequence

$$(6) \quad 0 \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$$

the connectedness of  $C$ , which imposes  $h^0(C, \mathcal{O}_C) = 1$ , Serre duality on  $C$  and the isomorphism  $\omega_C \simeq \mathcal{O}_C$ .

(ii) By the normalization exact sequence we have the inclusion  $H^0(C, \mathcal{O}_C(l)) \hookrightarrow \oplus_i H^0(C_i, \mathcal{O}_{C_i}(l))$  for any  $l \in \mathbb{Z}$ , where  $C_i$  are the irreducible components of  $C$  which are either a projective line or a non-singular conic lying in a plane. So for  $l = -1$ , we have  $h^0(\mathcal{O}_C(-1)) = 0$ . Now, applying Riemann-Roch theorem to  $C$  (see [11] p.83) yields  $h^0(C, \mathcal{O}_C(1)) = \deg(\mathcal{O}_C(1)) = 5$  since the intersection of  $C$  with a generic hyperplane has degree 5. Moreover since  $C$  is not contained in a hyperplane,  $h^0(\mathcal{I}_C(1)) = 0$ . So tensoring (6) by  $\mathcal{O}_X(1)$  and taking the long exact sequence gives the desired cancelations.

(iii) By a projective transformation, we can suppose that  $C$  is the curve whose components are: the lines  $[A_0A_1]$ ,  $[A_1A_2]$ ,  $[A_2A_3]$  and the conic given by an equation of the form  $Q = \alpha X_4^2 + \beta X_0X_3 + \gamma X_0X_4 + \delta X_3X_4$  (so that it meets  $A_0$  and  $A_3$ ) in the plane  $X_1 = 0 = X_2$  where  $A_0 = [1 : 0 : \dots : 0], \dots, A_3 = [0 : \dots : 1 : 0]$ . Then we can check that the quadrics given by union of hyperplanes  $Q_0 = X_0X_2$ ,  $Q_1 = X_1X_3$ ,  $Q_2 = X_1X_4$ ,  $Q_3 = X_2X_4$  and the quadric  $Q$  form a basis of the space of quadrics containing  $C$ ; so we have  $h^0(\mathcal{I}_C(2)) = 5$ . The

cancellation of  $h^2(X, \mathcal{I}_C(2))$  follows immediately from the long exact sequence associated to (6) tensorized by  $\mathcal{O}_X(2)$ . For  $h^1(\mathcal{I}_C(2))$ , we proceed as in [14, Proposition IV.1.2]: for a generic hyperplane  $H$ , letting  $\Gamma := C \cap H = \{P_1, \dots, P_5\}$  where the  $P_i$  are five points whose span is equal to  $H$  and  $S := H \cap X$  a smooth cubic surface, we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{I}_C(1) & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & \mathcal{O}_C(1) \longrightarrow 0 \\
 & & \downarrow H & & \downarrow H & & \downarrow H \\
 0 & \longrightarrow & \mathcal{I}_C(2) & \longrightarrow & \mathcal{O}_X(2) & \longrightarrow & \mathcal{O}_C(2) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{I}_\Gamma(2) & \longrightarrow & \mathcal{O}_S(2) & \longrightarrow & \bigoplus_{i=1}^5 k_{P_i} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

For any  $P_{i_0}$ , since  $\Gamma$  spans  $H$ , there is a plane for the form  $h_1 = \text{span}(P_{j_1}, P_{j_2}, P_{j_3})$  ( $j_i \in \{1, \dots, 5\} \setminus \{i_0\}$ ) that do not contains  $P_{i_0}$ . we can also choose a plane  $h_2$  containing the last point  $P_k$  ( $\{k\} = \{1, \dots, 5\} \setminus \{i_0, j_1, j_2, j_3\}$ ) and not containing  $P_{i_0}$ , then the union  $h_1 \cup h_2$  is a quadric of  $H$  that does not contain  $P_{i_0}$ . Thus  $H^0(S, \mathcal{O}_S(2)) \rightarrow \bigoplus_{i=1}^5 k_{P_i}$  is surjective, so  $h^1(S, \mathcal{I}_\Gamma(2)) = 0$ . This gives the surjectivity of  $H^1(X, \mathcal{I}_C(1)) \rightarrow H^1(X, \mathcal{I}_C(2))$  hence  $h^1(\mathcal{I}_C(2)) = 0$ .

(iv) The first terms of the local-to-global spectral sequence used to compute the groups  $\text{Ext}(\mathcal{I}_C, \mathcal{O}_X(-2))$  gives

$$\begin{aligned}
 (7) \quad 0 &\rightarrow H^1(X, \mathcal{O}_X(-2)) \rightarrow \text{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \rightarrow H^0(X, \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2))) \\
 &\rightarrow H^2(X, \mathcal{H}om(\mathcal{I}_C, \mathcal{O}_X(-2))).
 \end{aligned}$$

By [24, lemma 1],  $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq \det(\mathcal{N}_{C/X}) \otimes i^* \underbrace{\mathcal{O}_X(-2)}_{=\omega_X}$ , where the vector bundle

$\mathcal{N}_{C/X}$  on  $C$  is the normal bundle of the locally complete intersection subscheme  $C$  in  $X$  and  $i : C \hookrightarrow X$  the inclusion so  $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq i_* \omega_C$  the dualizing sheaf of  $C$ , hence  $H^0(X, \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2))) \simeq H^0(C, \omega_C)$ . But  $\omega_C \simeq \mathcal{O}_C$  and  $C$  is proper and connected so  $H^0(C, \omega_C) \simeq k$ . Next we have  $H^1(X, \mathcal{O}_X(-2)) = 0$  and  $\mathcal{H}om(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq \mathcal{O}_X(-2)$  ([24, lemma1]) so that  $H^2(X, \mathcal{H}om(\mathcal{I}_C, \mathcal{O}_X(-2))) \simeq H^2(X, \mathcal{O}_X(-2))$ . This last group being zero, we have  $\text{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq H^0(C, \mathcal{O}_C) \simeq k$ . It is thus clear that the generator of  $\text{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-2))$  generates  $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X(-2))$  at any point of  $C$ .  $\square$

By the Serre construction in codimension 2 (see for example [24]), we can associate, using Proposition 4.4 (iv), to a locally complete intersection curve  $C \subset X$  constructed as above, a rank 2 vector bundle on  $X$ .

**Proposition 4.5.** *For any curve  $C \subset X$  constructed as above, there is a unique auto-dual rank 2 vector bundle  $\mathcal{E}$  on  $X$  fitting in the exact sequence:*

$$(8) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}(1) \rightarrow \mathcal{I}_C(2) \rightarrow 0$$

given by any non zero element of  $\text{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-2)) \simeq k$ .

Vector bundles obtained by this method have the following properties which were proved in [17, Lemmas 2.1, 2.7, 2.8 and Proposition 2.6] for vector bundles constructed by the same method but starting from smooth normal elliptic quintic curves:

**Proposition 4.6.** *Let  $\mathcal{E}$  be a rank 2 vector bundle obtained from a singular linearly normal elliptic quintic curve applying Proposition 4.5. Then we have:*

- (i)  $h^0(\mathcal{E}(-1)) = 0$ ,  $h^0(\mathcal{E}) = 0$ ,  $h^0(\mathcal{E}(1)) = 6$  and  $h^i(\mathcal{E}(-1)) = 0 = h^i(\mathcal{E}(1)) \forall i \geq 1$ ;
- (ii)  $\mathcal{E}(1)$  is a stable vector bundle generated by its global sections so that the zero locus of any non zero global section  $s$  of  $\mathcal{E}(1)$  is a local complete intersection curve  $C_s$  on  $X$  with trivial dualizing sheaf, whose ideal sheaf satisfies all the equalities of Proposition 4.4 and fits in an exact sequence  $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}(1) \rightarrow \mathcal{I}_{C_s}(2) \rightarrow 0$ ;
- (iii)  $h^0(\mathcal{E} \otimes \mathcal{E}) = 1$ ,  $h^1(\mathcal{E} \otimes \mathcal{E}) = 5$ ,  $h^2(\mathcal{E} \otimes \mathcal{E}) = 0 = h^3(\mathcal{E} \otimes \mathcal{E})$ ;
- (iv)  $h^0(\mathcal{N}_{C/X}) = 10$ ,  $h^1(\mathcal{N}_{C/X}) = 0$ . In particular, the Hilbert scheme  $\mathrm{Hilb}_{5n}(X/k)$ , which parametrizes 1-dimensional subschemes of degree 5 and arithmetic genus 1 in  $X$ , is smooth of dimension 10 at the point  $[C]$ .
- (v) The morphism  $\mathbb{P}^5 \simeq \mathbb{P}(H^0(\mathcal{E}(1))) \rightarrow \mathrm{Hilb}_{5n}(X/k)$ , that associates to any non-zero global section of  $\mathcal{E}(1)$  the subscheme of  $X$  defined by zero locus, is injective.

*Proof.* (i) The computation of the dimension of these cohomology groups is immediate from (8) and Proposition 4.4 using long exact sequences with appropriate twists and Serre's duality.

(ii) We have seen that  $h^1(\mathcal{E}) = 0$ ,  $h^2(\mathcal{E}(-1)) = 0$  and  $h^3(\mathcal{E}(-2)) = h^0(\mathcal{E}) = 0$  so by Mumford-Castelnuovo criterion,  $\mathcal{E}(1)$  is generated by its global sections. There is no assumption on the characteristic of the base field in the arguments used in [17, Proposition 2.6] to prove the stability of  $\mathcal{E}(1)$  so the proof adapts to our setting.

The proofs of items (iii) and (iv) given in [17, Lemma 2.7] adapt in positive characteristic since it only uses the fact that stability of vector bundles implies their simplicity, Grothendieck-Riemann-Roch theorem and the fact that  $h^1(\mathcal{N}_{C/\mathbb{P}_k^4}(2)) = 0$  which is still true in our setting since the second proof of this fact given in [14, Proposition V.2.1] makes no use of the characteristic ( $\neq 2$ ) nor of a better regularity than local complete intersection. Item (v) is [17, Lemma 2.8] whose proof uses no assumption on the characteristic of the base field.  $\square$

Define the locally closed subscheme  $\mathcal{H}$  of  $\mathrm{Hilb}_{5n}(X/k)$ :

$$(9) \quad \mathcal{H} = \left\{ [Z] \in \mathrm{Hilb}_{5t}(X/k), \begin{array}{l} (i) \text{ } Z \text{ is a locally complete intersection of pure} \\ \text{dimension 1, (ii) } h^1(\mathcal{I}_Z) = 0 = h^0(\mathcal{I}_Z(1)) = h^1(\mathcal{I}_Z(1)) \text{ (hence } h^0(\mathcal{O}_Z) = 1), \\ \text{(iii) } h^1(\mathcal{I}_Z(2)) = 0 = h^2(\mathcal{I}_Z(2)) \text{ (hence } h^0(\mathcal{I}_Z(2)) = 5), \text{ (iv) } \omega_Z \simeq \mathcal{O}_Z \end{array} \right\}.$$

The 6-dimensional family of singular linearly normal elliptic quintic curves constructed in the previous section is contained in  $\mathcal{H}_s$ . Moreover, the subschemes parametrized by  $\mathcal{H}$  are connected (since  $h^0(\mathcal{O}_Z) = 1$ ) local complete intersection curves with trivial dualizing sheaf and whose ideal sheaves have all the properties needed to guarantee that the coherent sheaf arising from Serre's construction in codimension 2 (see Proposition 4.5) is a rank 2 vector bundle on  $X$  satisfying the properties of Proposition 4.6. In particular, by item (iv) of Proposition 4.6,  $\mathcal{H}$  is a smooth 10-dimensional quasi-projective  $k$ -scheme. So, using the pull-back  $\mathcal{Z}$  of the universal sheaf over  $\mathrm{Hilb}_{5n}(X/k)$  on  $\mathcal{H}$ , we can define an Abel-Jacobi morphism  $\phi_{\mathcal{Z}} : \mathcal{H} \rightarrow J(X)$ . We have the following theorem which proves that the condition (\*) of the introduction is satisfied by cubic threefolds:

**Theorem 4.7.**  $\phi_{\Gamma}$  is smooth and its general fiber is isomorphic to  $\mathbb{P}_k^5$ .

*Proof.* To prove the smoothness of  $\phi_{\mathcal{Z}}$ , we just have to see that the differential of the Abel-Jacobi morphism  $T\phi_{\mathcal{Z}} : T_{[C]}\mathcal{H} \simeq H^0(C, \mathcal{N}_{C/X}) \rightarrow T_{\phi_{\mathcal{Z}}([C])}J(X) \simeq H^2(X, \Omega_{X/k})$  is surjective for any  $[C] \in \mathcal{H}$ . To do so, we proceed as in [17, Theorem 5.6] trying to prove that the dual of this map  $(T\phi_{\mathcal{Z}})^* : H^2(X, \Omega_{X/k})^* \simeq H^1(X, \Omega_{X/k}^2) \rightarrow H^0(C, \mathcal{N}_{C/X})^*$  is injective. We use the technique of "tangent bundle sequence" following [33, Section 2]. It is presented there for a smooth subvariety  $C \hookrightarrow X$  and in characteristic zero but the arguments to prove

the following lemma make no use, for the subvariety  $C$ , of a greater regularity than local complete intersection (so that  $\mathcal{N}_{C/X}$  and  $\mathcal{N}_{C/\mathbb{P}_k^4}$  are vector bundles on  $C$  and we can use Serre duality) nor of the characteristic ( $\neq 2$ ). So  $(T\phi_Z)^*$  admits the following description:

**Lemma 4.8.** ([33, Lemma 2.8]). *The following diagram is commutative*

$$\begin{array}{ccc} H^0(X, \mathcal{N}_{X/\mathbb{P}_k^4} \otimes \omega_X) & \xrightarrow{R} & H^1(X, \Omega_{X/k}^2) \\ r_C \downarrow & & \downarrow (T\phi_Z)^* \\ H^0(C, \mathcal{N}_{X/\mathbb{P}_k^4} \otimes \omega_{X|C}) & \xrightarrow{\beta_C} & H^0(C, \mathcal{N}_{C/X})^* \end{array}$$

where  $r_C$  is the restriction map to  $C$ ,  $\beta_C$  is part of the exact sequence

$$\cdots \rightarrow H^0(C, \mathcal{N}_{C/\mathbb{P}_k^4} \otimes \omega_X) \rightarrow H^0(C, \mathcal{N}_{X/\mathbb{P}_k^4} \otimes \omega_{X|C}) \xrightarrow{\beta_C} H^1(C, \mathcal{N}_{C/X} \otimes \omega_X) \simeq H^0(C, \mathcal{N}_{C/X})^*$$

from the long exact sequence arising from the short exact sequence

$$0 \rightarrow \omega_X \otimes \mathcal{N}_{C/X} \rightarrow \omega_X \otimes \mathcal{N}_{C/\mathbb{P}_k^4} \rightarrow \omega_X \otimes \mathcal{N}_{X/\mathbb{P}_k^4}|_C \rightarrow 0$$

and  $R$  is the first connecting morphism in the long exact sequence associated to

$$0 \rightarrow \Omega_{X/k}^2 \otimes \mathcal{N}_{X/\mathbb{P}_k^4}^* \rightarrow \Omega_{\mathbb{P}_k^4/k}|_X \rightarrow \omega_X \rightarrow 0$$

coming from the exterior cube of the short exact sequence  $0 \rightarrow \mathcal{N}_{X/\mathbb{P}_k^4}^* \rightarrow \Omega_{\mathbb{P}_k^4/k}|_X \rightarrow \Omega_{X/k} \rightarrow 0$ .

Now,  $R$  is an isomorphism since Bott formula for  $\mathbb{P}_k^4$  imply the vanishing of  $H^0(X, \Omega_{\mathbb{P}_k^4/k|X}^3(3))$  and  $H^1(X, \Omega_{\mathbb{P}_k^4/k|X}^3(3))$ . The kernel of  $r_C$  is  $H^0(X, \mathcal{N}_{X/\mathbb{P}_k^4} \otimes \omega_X \otimes \mathcal{I}_{C/X}) = H^0(X, \mathcal{I}_{C/X}(1))$  which is 0 by assumption. As for the kernel of  $\beta_C$ , we have already seen in the proof of Proposition 4.6, that  $H^1(C, \mathcal{N}_{C/X}^*(2)) = H^0(C, \mathcal{N}_{C/X}(-2))$  is 0 since the second proof given in [14, Proposition V.2.1] for the vanishing of  $h^1(C, \mathcal{N}_{C/\mathbb{P}_k^4}^*(2)) = h^0(C, \mathcal{N}_{C/\mathbb{P}_k^4}(-2))$  makes no use of the characteristic  $\neq 2$  nor of a greater regularity than local complete intersection and  $\mathcal{N}_{C/X}(-2) \subset \mathcal{N}_{C/\mathbb{P}_k^4}(-2)$ ; so  $\beta_C$  is also injective. Hence  $T\phi_Z$  is surjective at all points of  $\mathcal{H}$  i.e.  $\phi_Z$  is smooth on  $\mathcal{H}$ .

Since  $\phi_Z$  is smooth, any nonempty closed fiber of  $\phi_Z$  is a disjoint union of smooth  $k$ -varieties. In fact it is the union of copies of  $\mathbb{P}_k^5$ : using the curve  $C$  parametrized by a point  $[C]$  of this fiber, we can construct a rank 2 vector bundle  $\mathcal{E}$  having the properties of Proposition 4.6; in particular  $\mathbb{P}(H^0(\mathcal{E}(1))) \simeq \mathbb{P}_k^5$  (4.6 (i)) and the inclusion of 4.6 (v)  $\mathbb{P}(H^0(\mathcal{E}(1))) \hookrightarrow \text{Hilb}_{5n}(X/k)$  has values in  $\mathcal{H}$  since, by 4.6 (ii), the curves defined by the zero locus of elements of  $\mathbb{P}(H^0(\mathcal{E}(1)))$  satisfy the conditions defining  $\mathcal{H}$ . Since a morphism from a projective space to an abelian variety is constant, we see that  $\phi_Z(\mathbb{P}(H^0(\mathcal{E}(1)))) = \phi_Z([C])$  i.e.  $\mathbb{P}_k^5 = \mathbb{P}(H^0(\mathcal{E}(1)))$  is included in the fiber  $\phi_Z^{-1}(\phi_Z([C]))$  and  $[C] \in \mathbb{P}_k^5 = \mathbb{P}(H^0(\mathcal{E}(1)))$  (by exact sequence 8). So every nonempty closed fiber of  $\phi_Z$  is the disjoint union of  $\mathbb{P}_k^5$ .

Now, let  $\pi : \mathfrak{X} \rightarrow S = \text{Spec}(R)$  be a smooth projective lifting of the smooth cubic  $X \subset \mathbb{P}_k^4$  to characteristic 0 over a DVR. Let us denote by  $K = \text{Frac}(R)$ , the generic point of  $S$  and  $s$  the closed point. We have also an abelian scheme over  $S$ ,  $\mathcal{J} = \text{Pic}^0(\mathcal{F}(\mathfrak{X})/S)$ , the relative Picard scheme of the relative Fano surface of  $\pi$ , whose geometric fibers are isomorphic to the intermediate jacobian of the corresponding smooth cubic hypersurfaces. Let  $\mathcal{H}(\mathfrak{X}/S)$  be the locally closed subscheme of the relative Hilbert scheme  $\text{Hilb}_{5t}(\mathfrak{X}/S)$  defined as

$$(10) \quad \left\{ \begin{array}{l} [Z] \in \text{Hilb}_{5t}(\mathfrak{X}/S), \text{ (i) } Z \text{ is a locally complete intersection of pure} \\ \text{dimension 1, (ii) } h^1(\mathcal{I}_Z) = 0 = h^0(\mathcal{I}_Z(1)) = h^1(\mathcal{I}_Z(1)) \text{ (hence } h^0(\mathcal{O}_Z) = 1), \\ \text{(iii) } h^1(\mathcal{I}_Z(2)) = 0 = h^2(\mathcal{I}_Z(2)) \text{ (hence } h^0(\mathcal{I}_Z(2)) = 5), \text{ (iv) } \omega_Z \simeq \mathcal{O}_Z \end{array} \right\}.$$

whose fiber over  $s$  is  $\mathcal{H}$  and fiber over  $K$  is the subscheme of the Hilbert scheme  $\text{Hilb}_{5n}(\mathfrak{X}_\eta/K)$  defined likewise which, according to [17, Corollary 5.5], is also smooth of dimension 10. It is easy to see that a singular linearly normal elliptic quintic curve, as those constructed in 4.2,

lifts in characteristic 0 over  $R$  so  $\mathcal{H}(\mathfrak{X}/S)$  has a  $S$ -point that we can use as a reference point to define a morphism  $\phi_{\mathcal{Z}_S} : \mathcal{H}(\mathfrak{X}/S) \rightarrow \mathcal{J}$  (using the universal family  $\mathcal{Z}_S$  of  $\mathrm{Hilb}_{5t}(\mathfrak{X}/S)$ ) inducing the Abel-Jacobi morphisms over  $s$  and  $K$ .

Let  $\Gamma_{\phi_{\mathcal{Z}_S}}$  be the closure of the graph of  $\phi_{\mathcal{Z}_S}$  in the  $S$ -projective scheme  $\mathrm{Hilb}_{5n}(\mathfrak{X}/S) \times_S \mathcal{J}$ . Then the second projection  $p_S : \Gamma_{\phi_{\mathcal{Z}_S}} \rightarrow \mathcal{J}$  yields a projective morphism inducing the Abel-Jacobi morphism  $\phi_{\mathcal{Z}_S}$  on a dense open subscheme of  $\Gamma_{\phi_{\mathcal{Z}_S}}$  which is surjective on  $S$ . By Stein factorization theorem, there is a proper  $S$ -scheme  $\mathcal{M}$  such that  $p_S$  factorizes as  $\Gamma_{\phi_{\mathcal{Z}_S}} \xrightarrow{\Phi} \mathcal{M} \xrightarrow{\Psi} \mathcal{J}$  with  $\Psi$  a finite morphism and  $\Phi$  a morphism with connected fibers. By work of Iliev, Markutchevich and Tikhomirov ([17, Theorem 5.6] and [15, Theorem 3.2]; see also Druel [9, Theorem 1]), the general fiber of  $p_K$  is  $\mathbb{P}_K^5$  so that  $\Psi_K : \mathcal{M}_K \rightarrow \mathcal{J}_K$  is an isomorphism. So  $\Psi$  is a birational morphism. Let us denote  $\sigma : \mathcal{J} \dashrightarrow \mathcal{M}$  the inverse map. Since  $\mathcal{J}$  is regular, the local ring of any codimension 1 point of  $\mathcal{J}$  is a DVR and since  $\mathcal{M}$  is proper,  $\sigma$  is defined on any codimension 1 point. In particular, it is defined at the generic point of  $J(X) = \mathcal{J}_s$ ; hence  $J(X)$  is birational to a component  $M_0$  of  $\mathcal{M}_s$ . Let us denote  $B = \Phi_s^{-1}(M_0)$ . Then  $p_s : B \rightarrow J(X)$  is dominant with general fiber isomorphic to  $\mathbb{P}^5$  and on a open subset of  $B$ , it is the Abel-Jacobi morphism given by the pull-back of the universal family of  $\mathrm{Hilb}_{5t}(X/k)$ .  $\square$

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